Problem Solving, Modeling and Local Conceptual Development

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Research reported here describes similarities and differences between (i) *modeling cycles* that students typically go through during sixty-to-ninety minute solutions to a class of problems that we refer to as *model-eliciting activities*, and (ii) *stages of development* that students typically go through during the "natural" development of constructs (conceptual systems, cognitive structures) that cognitive psychologists consider to be relevant to these specific problems. Examples of relevant constructs include those that underlie children's developing ways of thinking about fractions, ratios, rates, proportions, or other elementary-but-deep mathematical ideas. Thus, what we are investigating is that fact that, when problem solvers go through an iterative sequence of testing and revising cycles in order to develop productive models (or ways of thinking) about a given problem solving situation, and when the conceptual systems that are needed are similar to those that underlie important constructs in the school mathematics curriculum, then these modeling cycles often appear to be local or situated versions of the general stages of development that developmental psychologists and mathematics educators have observed over time periods of several years for the relevant mathematics constructs. Furthermore, the processes that contribute to *local conceptual development* in *model-eliciting activities* are similar in many respects to the processes that contribute to general cognitive development.

Applying principles from developmental psychology to problem solving—and vice versa—is a relatively new phenomenon in mathematics education (Lesh & Zawojewski, 1987; Zawojewski & Lesh, 2002). One implication of this approach is that we expect the mechanisms that contribute to general conceptual development to be useful to help explain students' problem solving processes in individual problem solving sessions (Lester & Kehle, 2002). Or conversely, mechanisms that are important in *local conceptual development sessions* should help explain the situated development of students' general reasoning capabilities (Harel & Lesh, 2002). Yet, when problem solving is interpreted as *local conceptual development*, many currently prevailing views about conceptual development need to be revised substantially (Lesh & Doerr, 2002). For example, it becomes clear that the development of powerful conceptual systems often is a great deal more situated, piecemeal, multi-dimensional, and unstable than has been suggested by Piaget-inspired ladder-like descriptions of cognitive development (DiSessa, 1988). Furthermore, the process of gradually sorting out and refining *unstable* conceptual systems tends to be quite different than the process of constructing (or assembling) *stable* conceptual systems (Lesh & Doerr, 1998). In particular, later in this journal, the article by Doerr, Lesh, Carmona & Haljmarson describes a variety of ways that *models and modeling perspectives* have required us to move significantly beyond *constructivist* ways of thinking about mathematics teaching, learning and problem solving. In the meantime, the primary aim of this article is to clarify the nature of *local conceptual development*—because situated development of powerful, sharable, and re-useable models is cornerstone concept underlying many of the most significant aspects of our *models and modeling perspective*, this phenomenon

FOR MODEL-ELICITING ACTIVITIES, PROBLEM SOLVING PROCESSES USUALLY INVOLVE MULTIPLE MODELING CYCLES

In general, the kind of problem solving situations that we emphasize are simulations of real life experiences where mathematical thinking is useful in the everyday lives of students, or their friends or families. In particular, many are middle school versions of the kind of *case studies* that are emphasized (for both instruction and assessment) in many of our nation's leading graduate schools in fields ranging from aeronautical engineering to business management—where leaders are being groomed for success in the *21^{st} century*. We are especially interested in these *case studies for kids* because one of the goals of our research is to clarify the nature of the most important mathematical understandings and abilities that provide foundations for success beyond school in a technology-based *age of information* (Lesh, Zawojewski, Carmona, 2002).

A distinguishing characteristic of the preceding *case studies for kids* is that the products that problem solvers produce generally involve much more than simply giving brief answers to well formulated questions. In fact, relevant solution processes usually involve much more than simply getting from givens to goals when the path is blocked (Lester & Kehle, 2001). That is, problem solvers produce conceptual tools that include explicit mathematical models for constructing, describing, or explaining mathematically significant systems. In other words, *case studies for kids* are *model-eliciting activities* (Lesh, Hole, Hoover, Kelly, & Post, 2000); and, the models that problem solvers produce include constructs and conceptual systems that are needed to make sense of the kind of complex systems that are ubiquitous in a technology-based *age of information* (Lesh, 2001).

Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media - which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s).

Models include both: (i) a conceptual system for describing or explaining the relevant mathematical objects, relationships, actions, patterns, and regularities that are attributed to the problem solving situation, and (ii) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals.

Mathematical models are distinct from other categories of models mainly because they focus on structural characteristics (rather than, for example, physical, biological, or artistic characteristics) of systems they describe.

Model development typically involves quantifying, organizing, systematizing, dimensionalizing, coordinatizing, and (in general) mathematizing objects, relations, operations, patterns, or rules that are attributed to the modeled system. Consequently, the development of sufficiently useful models typically requires a series of iterative "modeling cycles" where trial descriptions (constructions, explanations) are tested and revised repeatedly.

DIFFERENCES BETWEEN *MODEL-ELICITING ACTIVITIES* AND *TRADITIONAL WORD PROBLEMS*

One of the most significant characteristics that make *model-eliciting activities* different than most traditional textbook *word problems* is that, for the latter, the main thing that's problematic (beyond difficulties associated with computational skills that such problems generally emphasize) is that students must make meaning out of a symbolically described situation. Whereas, for *model-eliciting activities*, what's most problematic is that students must make productive symbolic descriptions of meaningful situations. That is, descriptions and explanations (or constructions) are not just relatively insignificant accompaniments to "answers." They ARE the most critical components of conceptual tools that need to be produced. Therefore, development processes generally involve a series of modeling cycles in which the problem solvers' ways of thinking about the "given" information (e.g., givens, goals, and available solution steps) need to be tested and revised iteratively.[1]

Another relevant characteristic that makes *model-eliciting activities* strikingly different than most traditional textbook word problems is that, in most real life situations where tools need to be developed, it's usually clear: (i) who needs the tool and (ii) why, or for what purpose, the tool is needed. For example, if the goal were to build a boat so that a group of students can cross a river, then the product that the students need to produce is the boat

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[1] Traditionally, in mathematics education research & development, problem solving has been defined as *getting from givens to goals when the path is not immediately obvious or it is blocked*; and heuristics have been conceived to be answers to the question: *What can you do when you are stuck?* But, when attention shifts toward on *model-eliciting activities* in which a series of interpretation cycles are required to produce adequate ways of thinking (about givens and goals), then the essence of problem solving involves finding ways to interpret these situations mathematically. Therefore, heuristics and strategies that tend to be most useful focus on helping students find productive ways to adapt, modify, and refine ideas that they DO HAVE, rather than being preoccupied with finding ways to help them be more effective when they are stuck (e.g., in puzzles and games when they have no relevant ideas, or when no substantive constructs appear to be relevant). In general, for problems whose solutions involve multiple modeling cycles, the kinds of heuristics and strategies that are most useful tend to be quite different than those that have been emphasized in traditional problems in which the solutions typically involve only a single interpretation cycle to make sense of the situation.

(whose design is unspecified); and, the test of whether the design is adequate is based on the criteria of crossing the river safely. In other words, the purpose provides an "end in view" (Dewey, J. (Archambault,1964)) that provides a way for problem solvers to assess the adequacy of their current way of thinking about the nature of product design —or strengths and weaknesses of alternative designs (English & Lesh, 2002).

A third distinctive characteristic of *model-eliciting activities* is that it seldom makes sense to devote much effort toward the development of tools unless the goal is to deal with more than a single isolated situation. That is, tool development is worthwhile mainly when the product is intended to be <u>sharable</u> (with other people), <u>re-usable</u> (in other situations), and <u>modifiable</u> (for other purposes). Consequently, some of the most important ways that tools are tested are directly related to these characteristics. In particular, tool development is inherently a social activity (Middleton, Lesh & Heger, 2002) because the tools need to be sharable with others - as well as being useful to yourself at some later time (or in new situations).

MULTIPLE MODELING CYCLES & LOCAL CONCEPTUAL DEVELOPMENT

Sometimes, we design *model-eliciting activities* so that the constructs and conceptual systems that problem solvers need to develop involve the same operational/relational schemes (or cognitive structures) that developmental psychologists have investigated for children's mathematical judgments about the underlying ideas that are involved in the problem—e.g., fractions, ratios, rates, proportions. Consequently, when students significantly extend, revise, or refine their ways of thinking about these constructs during a single relatively brief problem solving session, we refer to these problem solving sessions as *local conceptual development sessions*. … One reason why this term has proven to be especially appropriate is because the modeling cycles that students go through during sixty-to-ninety minute problem solving sessions often appear to be remarkably similar to the stages of development that developmental psychologists have observed, for the same constructs, over time periods of several years (Lesh & Kaput, 1988). Consequently, it is possible to directly observe *processes* that lead to extensions, refinements, revisions, or adaptations in students' ways of thinking.

Examples of several *local conceptual development sessions* are described in the next section of this article. All of the problems in the next section involve some type of proportional reasoning (or "scaling up"). So, for each problem, we briefly describe typical transcript in which the modeling cycles that students go through are similar to stages of development that have been described by Piaget and others concerning children's developing ways of thinking about situations that mathematicians might characterize using the proportion A/B=C/D (Inhelder & Piaget, 1958).

According to Piaget, the most essential characteristic of proportional reasoning to be that it involves a relationship between two relationships (A/B=C/D). That is, the relationship between A and B is compared to the relationship between C and D (Lesh, Post, & Behr, 1989). Also according to Piaget and other development psychologists (Piaget, Inhelder, & Szeminska, 1964), children's proportional reasoning capabilities develop through the following basic stages (Behr, Post, & Lesh, 1982):

Summary of Piaget's Stages of Development for Proportional Reasoning

<u>Stage #1:</u> Early reasoning about proportions tends to be based on only a salient subset of information that is available; and, quantities or relationships that are emphasized tend to be those that are easiest to point to directly. In their primitive responses to Piaget's proportional reasoning activities, students tend to ignore part of the relevant data. For example, in a balance beam task, students may notice only the size of the weights on each arm, but ignore the distance of the weights from the fulcrum. Therefore, reasoning tends to involve qualitative judgments (A<B) about counts, lengths, areas, or other types of quantities that Piaget considered to be based on only *concrete operational reasoning* (Inhelder & Piaget, 1958).

<u>Stage #2:</u> In some of the earliest situations in which students go beyond reasoning about <u>relationships between simple quantities</u> to be able to reason about <u>relationships between relationships</u>, the relevant relationships are considered to be additive (A-B=C-D) rather than multiplicative (A/B=C/D)

in nature. That is, the “difference” between A and B is compared to the “difference” between C and D (because these “difference” correspond to quantities that can be perceived directly). For example, if a student is shown a 2X3 rectangle and is asked to "enlarge it", a correct response is often given by "doubling" to make a 4X6 rectangle. However, if the request is then made to "enlarge it again so that the base will be 9", the same students often draw a 7X9 rectangle - adding 3 to both sides of the 4X6 rectangle.

Stage #3: In other early situations where students begin to reason about relationships between relationships, their reasoning is based on pattern recognition and replication. For example, if youngsters are given a simple table of values, like the one shown below, then they may be able to think about proportions by noticing a pattern which they can then apply to discover an unknown value.

A candy store sells 2 pieces of candy for 8 cents.
How much does 6 pieces of candy cost?

- 2 pieces for 8 cents
- 4 pieces for 16 cents
- 6 pieces for 24 cents
- 8 pieces for {?} cents

note: Some researchers call the preceding strategy a "build up" strategy (e.g., Hart, 1984; Karplus & Peterson, 1970, Inhelder & Piaget, 1958). According to Piaget, this type of reasoning does not necessarily represent "true proportional reasoning" because, to answer such problems correctly, students do not necessarily have to be aware of the reversibility of the relevant operations. That is, if a change is made in one of the four variables in a proportion, the student should be able to compensate by changing one of the remaining variables.

Stage #4: Reasoning is based on true multiplicative proportions. That is, it involves a "second-order" relationship between two relatioships; and, the first-order relationships are considered to be multiplicative in nature.

EXAMPLES OF LOCAL CONCEPTUAL DEVELOPMENT

This section briefly describes three *local conceptual development sessions* for three different *model-eliciting activities*: *The Sears Catalog Problem* (Lesh & Kaput, 1988), *The Big Foot Problem* (Lesh & Doerr, 2002b), and *The Quilt Problem* (Lesh & Carmona, 2002). All of these examples involve problems and transcripts that have been analyzed in isolation in other articles that we’ve published in the past. Our purpose here is to focus on trends that become salient only when similarities and differences are examined across several tasks and transcripts. Then, this cross-transcript analysis will set the stage for the analysis of a new transcript in the main results section of this paper. More complete descriptions of these transcripts can be found in the preceding publications. Or, complete transcripts can be downloaded from: http://tcct.soe.purdue.edu/library/

For each transcript that we describe in this section, the solution was generated by a group of three middle school students who were videotaped while they were working in situations that were simulations of situations that might reasonably occur in their everyday lives. Each transcript involved a different group of students. But, in each case, the students were enrolled in remedial mathematics classes because of their poor records of performance in past school experiences. Also, in each case, the students were from schools in large urban school districts that served predominantly minority and disadvantaged populations. The students worked on the problems during extended class periods that lasted approximately 90 minutes; and, to ensure that the students would be familiar with the contexts in which the problems occurred, their teachers typically ask them to read and discuss a relevant article from a *math-rich newspaper* that our research team produced. One goal of these “warm-up exercises” was to encourage students to engage their “real heads” (not just their “school heads”) when they attempted to make sense of the problems and situations. Consequently, teachers usually discussed the newspaper articles on the day before students are expected to begin to work on the related problem. Then, when the problem is introduced, the students tended to waste less time getting oriented to the situations.

(a) *The Sears Catalog Problem*: To introduce this problem, the students were given four resource documents: (i) a local newspaper from ten years ago, (ii) a ten-year-old "back to school" catalog for a local department store (Sears), (iii) a current local newspaper, and (iv) a current "back to school" catalog from the same department store. To familiarize students with the context of the problem, a math-rich newspaper article was proved and the accompanying "warm-up" questions were about the "back to school buying power" of $200 at the time that the old newspaper had been published. The goal of the problem was for the students to produce a new newspaper article telling readers (other students) how much money would be needed to have the same "back to school buying power" today that $200 had ten years earlier.

A Brief Summary of a Sears Catalog Transcript

Interpretation #1: The first interpretation of the problem was based on what Piaget calls "additive pre-proportional reasoning" (Piaget & Inhelder, 1956) Judgments were based on only a biased subset of relevant information. That is, without much reflection about which items to consider, the group began to calculate "price differences" by subtracting using pairs of old and new items that were assumed to be comparable. However, only a few items were considered, which were simply the items that were noticed first; and, no apparent thought was given to how these subtracted differences might allow a predictions to be made about future buying power. …Later, after calculating several price differences (apparently, in the hope of discovering some sort of "pattern"), the tediousness of computation prompted the students to consider *Why are we doing this anyway?* Then, perhaps in an attempt to answer this question, they began trying to determine a sensible list of items that a "typical" student might want to buy.

A Brief (and Foolish) Transition Interpretation: A second brief re-conceptualization of the problem was based on the notion that, because ten years had passed, perhaps items should cost ten times as much! … Although

this "brainstorm" was quickly recognized to be foolish, it appeared to serve several useful functions: (i) It introduced a (primitive) multiplicative way to think about relationships between "old prices" and "new prices." (ii) It re-focused attention on the overall goal of finding a way to determine price increases for a representative group of items—or for any given item.

In general, students' early interpretations were characterized by repeatedly "losing the forest because of the trees"—or vice versa. For example, whereas the first conceptualization lost sight of the overall goal when attention was focused on details (individual subtractive differences), the second conceptualization ignored details when attention was focused on the relationship between salaries. For example, the students failed to notice that prices definitely DID NOT increase by a factor of 10 over a ten year period.

Interpretation #2: The students' second major re-interpretation of the problem was based on what Piaget called 'pattern recognition and replication" type of pre-proportional reasoning (Inhelder & Piaget, 1958). That is, the students noticed that the price increases for several items were approximately a factor of 2. So, they guessed that the "new buying power" also should increase by a factor of "about two." … The students clearly recognized that this second interpretation showed real promise. However, the new clarity of thought that it introduced also enabled them to notice that (for example) some items actually decreased in price - even though most of the prices increased. So, the group still recognized inadequacies of their current ways of thinking!

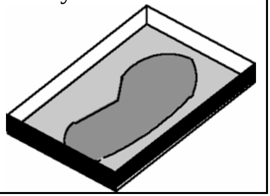
Interpretation #3: The students' third major re-conceptualization went beyond simply pattern repetition to thinking that involved proportional reasoning much more clearly—and didn't involve a simple whole number ratio. It also was based on a more sophisticated way of dealing with the difficulty that not all items increased by the same amount - and that some items actually decreased in price. That is, the students used their calculators to calculated what they referred to as "percent increases" (which were actually just simple multiplicative factor that expressed the relationship between an old price and a new price for each item). For example, they found that the sneakers went up by a factor of 2.75; the jeans went up by a factor of 2.25; and, the backpacks went up by a factor of 1.90. Then, these "percent increases" were AVERAGED—even though it was clear that the students were becoming increasingly worried about the facts such as: (i) not all items went up by the same factor, (ii) the prices of some items (e.g., calculators) actually decreased, (iii) there were disagreements about which items to include or ignore from their list of possible items to buy, and (iv) it sometimes wasn't clear that old and new items were actually comparable.

Interpretation #4: Interpretation #4 explicitly resolved the sampling issue! Also, for the first time, the students actually wrote a (crude) mathematical proportion of the form "A is to B as C is to D" —where the values for A and B were based on sums of prices for a number of "typical" items. Also, whereas interpretation #3 involved finding the average of a collection of "percent increases," interpretation #4 involved finding the "percent increase" for a collection of sums. But, perhaps the most significant insight associated with this new way of was that the students noticed that, after a sufficient number of items had been included in the sum, the factor that they were calling the "percent increase" was not affected much by the addition of more items. In other words, they noticed an intuitive sort of *law or large numbers*.

(b) *The Big Foot Problem*. During the class period before the *Big Foot Problem* was given to the students, the whole class discussed a newspaper article about Tom Brown, the famous tracker, who often works for police to find lost people or to track down criminals (Brown, 1978). Tom Brown is a real person who lives in New Jersey; and, when he was a young boy, an old Apache "grandfather" taught him to track and live in wild country with no tools or food except what he could make for himself using things he could find. Now, Tom is like the famous detective, Sherlock Holmes. He can find tracks where other people cannot see them; and, just by looking at tracks, he is able to figure out about how big the person is, about how heavy they are, about how fast they are walking or running, and a great many other things. … During the first two minutes of the problem solving session , the observer read the following problem statement to the students. Then, she showed the students a flat box (2"x2'x2') that was filled with a 1" layer of dried mud in which a large (size 24 Reebok cross training shoe) footprint could be seen.

Statement of the Big Footprint Problem

Early this morning, the police discovered that, sometime late last night, some nice people rebuilt the old brick drinking fountain in the park. The mayor would like to thank the people who did it. But, nobody saw who it was. All the police could find were lots of footprints. … You've been given a box (shown below) showing one of the footprints. The person who made this footprint seems to be very big. But, to find this person and his or her friends, it would help if we could figure out how big the person really is?



Your job is to make a "HOW TO" TOOL KIT that the police can use to figure out how big people are - just by looking at their footprints. Your tool kit should work for footprints like the one that is shown here. But it also should work for other footprints.

A Brief Summary of a Big Foot Transcript

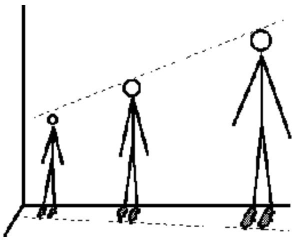
Interpretation #1– based on qualitative reasoning: For the first 8 minutes of the session, the students used only global qualitative judgements about the size of footprints for people of different size and sex—or for people wearing different types of shoes. e.g., … *Wow! This guy's huge. … You know any girls that big?!! … Those're Nike's. - The tread's just like mine.*

Interpretation #2 – based on additive reasoning: One student put his foot next to the footprint. Then, he used two fingers to mark the distance between the toe of his shoe and the toe of the footprint. Finally, he moved his hand to imagine moving the distance between his fingers to the top of his head. This allowed him to estimate that the height of the person who made the footprint. But, instead of thinking in terms of multiplicative proportions (A/B=C/D), using this approach, the students were using additive differences. That is,if one footprint is 6" longer than another one, then the heights also were guessed to be 6" different.

Note: At this point in the session, the students' thinking was quite unstable. For example, nobody noticed that one student's estimate was quite different than another's; and, predictions that didn't make sense were simply ignored. … Gradually, as predictions become more precise, differences among predictions began to be noticed; and, attention began to focus on answers that didn't make sense. Nonetheless, "errors" generally were assumed to result from not doing procedure carefully – rather than from not thinking in productive ways.

Interpretation #3 – based on primitive multiplicative reasoning: Here, reasoning was based on the notion of being "twice as big". That is, if my shoe is twice as big as yours, then I'd be predicted to be twice as tall as you."

Interpretation #4 – based on pattern recognition: Here, the students used a kind of concrete graphing approach to focus on trends across a sequence of measurements. That is, they lined up against a wall and used footprint-to-footprint comparisons to make estimates about height-to-height relationships as illustrated in the diagram shown here. … This way of thinking was based on the implicit assumption that the trends should be LINEAR—which meant that the relevant relationships were unconsciously treated as being multiplicative. The students said:



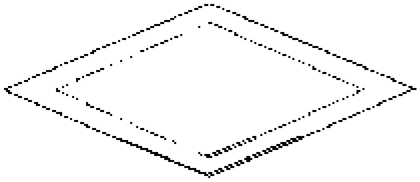
> *Here, try this… Line up at the wall… Put your heels here against the wall…. Ben, stand here. Frank, stand here…. I'll stand here 'cause I'm about the same (size) as Ben. {She points to a point between Ben and Frank that's somewhat closer to Ben}… {pause} … Now, where should this guy be? - Hmmm. {She sweeps her arm to trace a line passing just in front of their toes.}… {pause} … Over there, I think. - {long pause}… Ok. So, where's this guy stand? … About here. {She points to a position where the toes of everyone's shoes would line up in a straight line.}*

Note: At this point in the session, all three students were working together to measure heights, and the measurements were getting to be much more precise and accurate than earlier in the session.

<u>Interpretation #5:</u> By the end of the session, the students were being VERY explicit about comparing footprints-to-height. That is, they estimated that: Height is about six times the size of the footprint. For example, they say: *Everybody's a six footer! (referring to six of their own feet.)*

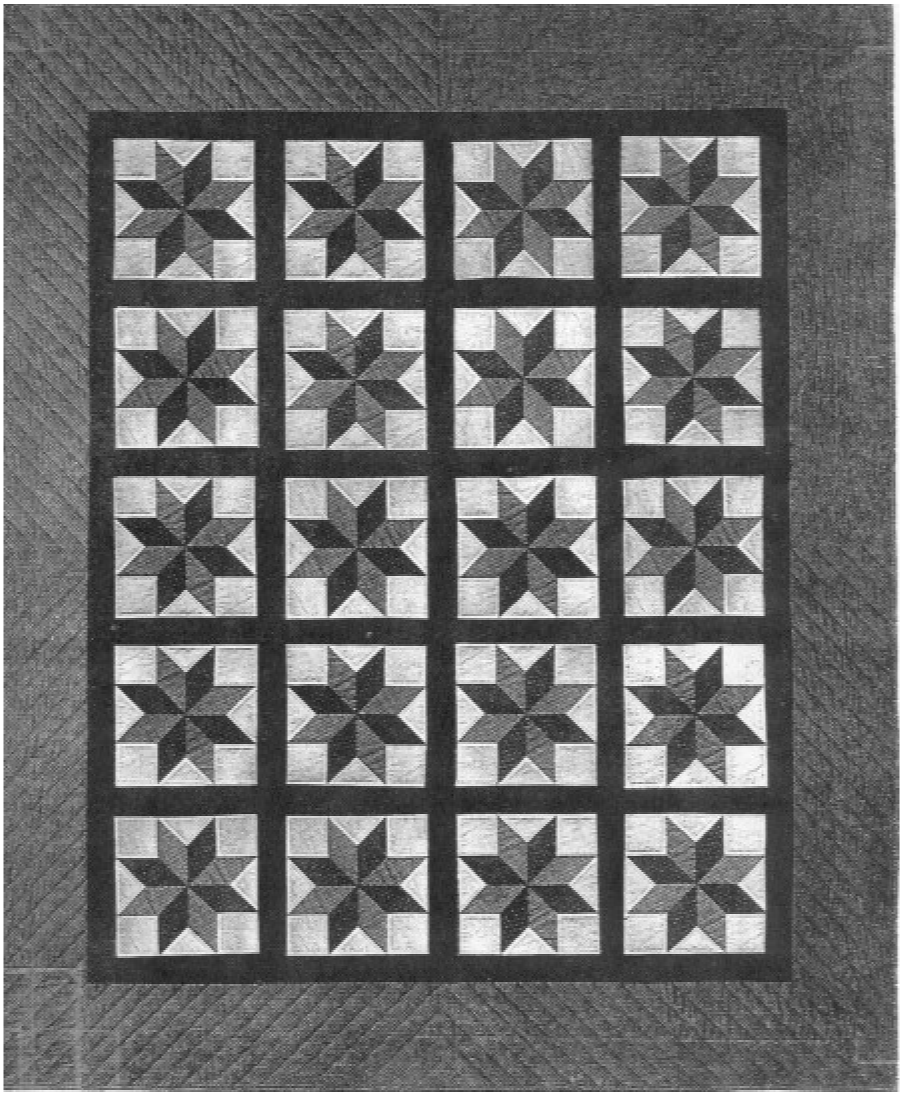
(c) *The Quilt Problem*: During the class period before the *Quilt Problem* was given, the whole class discussed a newspaper article about a local quilting club.

The article showed photographs of several different kinds of quilts; and, it also described how club members made the quilts using patterns and templates like the one shown here for a diamond shaped piece.



The problem statement described how quilt club members sometimes had difficulties when they tried to use photographs to make templates that were exactly the right size and shape to make quilts that club members found in books, newspapers, and magazines. So, the job for the students was to write a letter that did two things for the members of the quilting club. (a) First, the letter should describe procedures for making template pieces that were exactly the right size and shape for any quilt whose photograph they might find. (b) Second, the letter should include examples of how to follow their procedures by making templates for each of the pieces of the quilt that's shown on the following page.

note: The quilt that's shown was for a double bed. So, the finished size needed to be approximately 78" by 93".



An Example Photograph of a Quilt

A Brief Summary of a Quilt Transcript

Interpretation #1a - Focus on a "Scaling Factor" between the Two Shapes-as-a-Whole – Without Worrying about the Sizes of Individual Pieces. In this transcript, it's significant to mention that the students whose work is reported had a significant amount of past experience working on other *model-eliciting activities* that involved scaling-up—or proportional reasoning. In particular, three weeks earlier, these same three students had worked on the *Big Foot Problem.* Therefore, it was natural for them to begin the *Quilt Problem* by trying to use a process similar to the one that they ended up using earlier. Consequently, they began this session by trying to find a single "scaling factor" that could be used to stretch the picture of a quilt to make it the size of a real 78" by 94" quilt—without worrying about the sizes of individual pieces. … Two difficulties arose when they tried to use this approach; and, the result of these two difficulties led the students to produce results that they didn't consider to make sense. First, their ruler-based measurements were not very accurate, and they sometimes were not even correct (e.g., 4 5/8 inches was punched into their calculator as 4.5 inches). Second, the scaling factor that their calculators produced was thought of as a being very messy number (78/4.5=17.33333…) that didn't seem sensible. (note: The students themselves were only aware of the second difficulty.)

Interpretation #1b: Scale-up Individual Pieces within a Whole. In attempts to avoid "messy numbers" the students tried to find "scaling-up factors" based on individual pieces of the quilt—rather than being based on the quilt-as-a-whole. (note: In their earlier attempts, there was no evidence that the whole was viewed as being aggregated from its parts—nor that the same scaling factor should apply uniformly to both the whole and to its parts.) … Using this new piece-by-piece approach, new difficulties arose because the scaled-up pieces didn't fit together nicely. Thus, the students started trying to measure more carefully (e.g, measuring to the nearest sixteenths rather than eighths of an inch). In spite of these attempts, however, they found that more accurate measurements didn't lead to better results. That is, the scaled-up pieces still didn't fit together nicely; and, the "scaling-up factors" that they found also continued to be "messy numbers" that they didn't consider to make sense.

Interpretation #2: Map from One Unit of Measure to Another Within Two Similar Situations. Using interpretation #1b, the students believed that their "errors" (i.e., "messy numbers") must have resulted from measurement errors or from calculation errors. So, they tried to measure more accurately and consistently. In particular, they began to measure all of the pieces by counting sixteenths-of-an-inch markings on their rulers. (note: The students still had not spoken of what units they were using for their measurements. For example, *three-fourths of an inch* was read simply as *three-fourths*.) As a result of measuring in this way, they stumbled upon a lucky similarity between the picture and the real quilt. That is, the side of the picture that was supposed to be 93 inches turned out to be 91 (sixteenths-of-an-inch) marks on the ruler; and, similarly, the side that was supposed to be 78 inches turned out to be 76 (sixteenths-of-an-inch markings). So, in both of these instances, they noticed that the difference was 2. This lucky correspondence allowed them to adopt a procedure that essentially translated one sixteenth of an inch (for the picture) to one inch (for the real quilt). This new way of thinking focused on the relationships between the units that were used to measure the picture and the quilt—rather than focusing on relationships between small and large pieces of the quilt.

Intepretation #3a: Begin to Focus on Part-Part Relationships Within a Give Quilt (or Picture). Partly because the students were becoming increasingly concerned about how the pieces were or were not fitting together, they begin to pay more attention to quantitative relationships involving part-part comparisons within a given quilt (or picture). For example, they began to talk about the fact that the sides of the diamonds should be the same as the sides of the small squares—and about the fact that the "big squares in the quilt" should be the sum of several small pieces (i.e., the small squares and triangles). Thus, they sometimes changed "apparent measurements" in order to force these measurements to give results that they knew should be true (in order to make the pieces to fit together properly).

Interpretation #3b: Explicitly Compare Wholes and Sums-of-Parts—and also Comparing Measurements in Two Dimensions (Height and Width). Here, for the first time, the students explicitly stated that when the whole and the parts are scaled-up, the sum of the scaled-parts should be equal to the scaled-up whole. But, no matter now accurately they tried to measure, and no matter how carefully they used their calculators, the sum of the scaled-up parts did not equal the expected scaled-up whole. For example, when they scaled-up the big square (in the picture of the quilt) they got a different result than when they scaled up the small squares and the small triangles.

<u>Interpretation #4: Focus on Part-part Comparisons Based on Shapes with Shared Components (Sides).</u> Once the students were convinced that they had found a way to scale up one of the pieces (i.e., the small square), they measured other pieces using this small square as a unit. Instead of applying a single "scaling factor" to each piece of the quilt, they simply scaled up a single piece (i.g., the small square); then they measured all other pieces using this single piece. For example, they recognized that: (i) the little triangles should be half of the little squares, (ii) the triangles had two short sides and one longer side, (iii) the short sides of the triangles should be the same as the sides of the "kite" shapes. Using this process, they were willing to "adjust" the size of the whole quilt in order to make the parts fit together properly. In other words, they began to <u>deduce</u> lengths rather than simply <u>measuring</u> lengths; and, they were willing to change what they "saw" (based on direct measurements) in order to force these measurements to fit what they "understood" (e.g., that the sum of the parts should equal the whole).

Note: At this point in the session, the students began to act as if they believed they had completed the assigned task. But, when they announced to their teacher that *"We're done!"* she reminded them that part of their goal was make a template for each of the parts of the quilt shown in the picture. … So, the students went back to work.

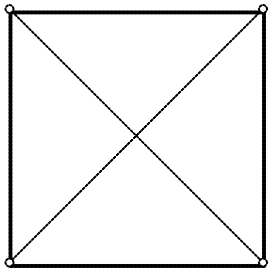
<u>Interpretation #5a: Part-Part Comparisons Based on Angles as well as Lengths.</u> Finally, because the students were increasingly precise concerning relationships among pieces, and because they were concerned that the "kite shaped pieces" weren't turning out quite right, they began to focus on angles as well as the lengths of sides. For example, because the small triangle was half of a small square, they noticed that these triangles must have a "square angle" (i.e., a right angle). Therefore, once they created templates for the triangles and the small squares, they recognized that they could use these templates to figure out the templates for the "kites."

<u>Intepretation #5b: Coordinate all of the Previous Quantitative Relationships (Whole-Whole, Part-Whole, and Part-Part—Concerning both Angles and Lengths).</u> Once the students created two comparable pieces for the small quilt and the real quilt, all of their other quantitative comparisons focused on part-whole and part-part comparisons—including relationships involving both angles and shared sides and lengths. In particular, because they now had a much more clear understanding that the sum of the parts should be equal to the whole (big square), they checked carefully to make sure that: (i) the templates for all of the pieces fit together to make a whole (i.e., big square), and (ii) the big squares plus their borders fit together to make a quilt that was the correct size. … To make all of the pieces fit together properly, the students went through several construct>test>revise cycles.

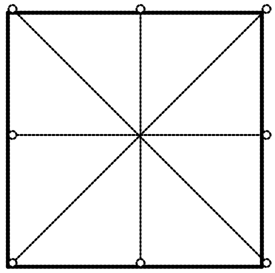
<u>Interpretation #6: A Two-stage Process focusing on Symmetry Relationships – using the Big Square (Rather than the Little Square) as the Basic Unit to Be Scaled-Up.</u> Here, for the first time, the sizes and shapes of the smaller pieces were derived from the shape of the large square (and the quilt-as-a-whole) rather than deriving the shape of the large square from the sizes and shapes of the smaller pieces. That is, the students final way of thinking about the quilt problem used a two-step process: (i) First, determine the size

of big squares, the borders, the thin strips, and the quilt-as-a-whole. (ii) Then, use paper folding (as illustrated below) to determine the sizes and shapes of the small pieces within the large squares.

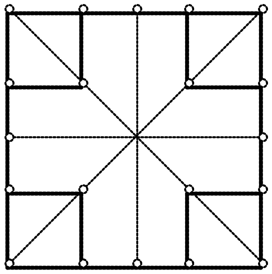
1. Start with a 12”x12”paper square, and fold it along the diagonals.



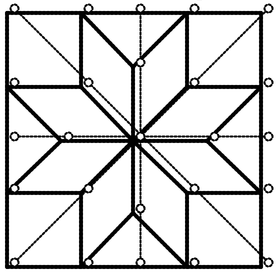
2. Fold the paper squares vertically and horizontally.



3. Measure one 3”x3” square in each corner of the bigger square.



4. Cut 4 half-squares. Each is half of one of the small squares



CONCLUSIONS RELATED TO LOCAL CONCEPTUAL DEVELOPMENT

All three transcripts described in the preceding section illustrate the fact that, when students are able to make sense of situations based on extensions of their own personal knowledge and experiences, and when they express their ways of thinking in forms that they themselves are able test and refine repeatedly, they often invent (or significantly modify, extend, or revise) mathematical constructs that are considerably more sophisticated than those that they seemed unable to comprehend during past histories of failure in situations centered around traditional textbooks, tests, and teaching. In particular, our research has shown consistently that youngsters who are among the least advantaged often invent more powerful ideas than anybody has ever dared to try to teach to them.

The transcript for the *Sears Catalog Problem* (Lesh & Kaput, 1988) was the first one where we began to recognize the striking similarities that often exist between the modeling cycles that students go through during *model-eliciting activities* and the stages of development that developmental psychologists have observed for the relevant mathematical ideas (Lesh & Kaput, 1988). This insight first became clear in the videotape analysis laboratory for a series of projects that are known collectively as *The Rational Number Project* (Post, Behr, & Lesh, 1982). One table in the lab contained drafts of a paper that we were writing about Piaget’s description of the development of children’s proportional reasoning concepts (Lesh, Post, & Behr, 1989); and, a second table contained drafts of a paper that we were writing about modeling cycles that problem solvers typically go through during solutions to the *Sears Catalog Problem*. … What we noticed was that the stages described on one table were nearly identical to the modeling cycles described on the second table.

The preceding observation also applies to the *Big Foot Transcript* and the *Quilt Transcript* reported in the preceding section. … One reason why this insight has proved to be significant is because, when students move through a series of Piaget-style stages during a single problem solving session, it is possible to go beyond investigating *states* of development to directly observe *processes* and *paths* that lead from one state to another. Also, because model development can be induced by putting students in situations where they express their thinking in ways in forms that they themselves can test and revise, and because students themselves determine directions for improvement, it is possible to go beyond investigating *typical* development in *natural* environments to also examine *induced* development within *carefully controlled* and *mathematically enriched* environments (Lesh, Hole, Hoover, Kelly & Post, 2000). In other words, it is possible to go beyond investigating the development of *general cognitive structures* that emerge during periods of *major cognitive reorganizations* (when children are approximately 6-7 or at 11-12 years of age) to focus on *transition periods* of development (before/between/following the ages of 7 and 12) for *powerful and highly specialized mathematical constructs* that seldom develop beyond primitive stages unless *artificially enriched mathematical experiences* are provided.

Because all three of the transcripts described in the preceding section involved some type of proportional reasoning (or "scaling-up"), an examination of similarities and differences across all three transcripts revealed several important facts that are not apparent in analyses of any single transcript. For example, a student's apparent level of development not only changes significantly during the course of solving each individual model-eliciting activity, it also changes significantly as the student moves from one activity to others that are structurally similar. Thus, just because a student's way of thinking about one problem has moved from stage N to stage N+n (n=1,2,3,…), this does not guarantee that he or she will immediately function at this highest level when beginning to work on another problem that's structurally similar. For example, initial conceptualizations of the second problem may revert back to thinking at level N or N-1. … This fact is especially apparent in the complete transcripts that are available to be downloaded for the *Big Foot Problem*. This is because transcripts are given in which a single group of students worked on a series of structurally similar model-eliciting activities.

The preceding observation could be interpreted as an instance of Vygotsky's concept of zones of proximal development (Vygotsky, 1978; Wertch, 1985). That is, at any given moment in any given situation, a student may have accessible a range of conceptual systems that could be engaged; and, which one DOES get engaged depends on a variety of factors that might include: guidance from an adult or peers, which among a variety of representational systems (or other conceptual tools) happen to be employed.

An implication of the preceding observations is that, when students begin to work on a *model-eliciting activity*, relevant constructs almost always should be expected to be at some intermediate stage of development. That is, they be expected to be "completely undeveloped" (so that they need to be constructed beginning with a "blank slate") nor should they should be expected to be "completely mastered." In general, the challenge for students is to extend, revise, reorganize, refine, modifying, or adapt constructs that they DO have—not simply to assemble constructs that are completely new.

Another related observation is that, even though the *Sears Catalog Problem*, the *Big Foot Problem*, and the *Quilt Problem* all involve some type of proportional reasoning, the situations also are quite different. For example, the *Sears Catalog Problem* gives an overwhelming amount of information that must be filtered, weighted, and organized in some way; whereas, the *Big Foot Problem* gives very little information directly; most needs to be generated. Consequently, for these and other reasons, each of the three problems involves not only ideas about "scaling-up" but their solutions also draw upon some other fundamentally different categories of ideas. For example, in the *Sears Catalog Transcript*, the students needed to develop some very important thinking about sampling and variability. Or, in the *Big Foot Transcript*, they introduced some important thinking about graphing, as well as about sampling and variability. Similarly, for the *Quilt Transcript*, the students developed a series of progressively more sophisticated ways of thinking about relevant units of measure. Consequently, the modeling cycles that students went through for the *Quilt Problem* looked more like the stages of development that Piaget and others have described for measurement concepts—rather than stages that have been described for proportional reasoning.

An important point to notice about the preceding observations is that it is seldom possible for problem solvers to give an adequate description of a non-trivial "real life" situation using only ideas from a single isolated topic area or discipline. Most solutions inherently involve integrating ideas from a variety topic areas and disciplines. Consequently, a large part of what it means to understand ideas in any given topic area or discipline involves establishing relationships with ideas in other topic areas and disciplines.

To help clarify the meaning of preceding observations, the final sections of this paper will describe a transcript for a forth *model-eliciting activity* that focuses on ideas in another topic area that is closely related to ideas about proportional reasoning and scaling-up. The topic area is projective geometry; the problem is called the *Shadows Problem*; and, the main point is that Piagetian stages of development are not the only kind that are useful for describing modeling cycles during *model-eliciting activities*. For example, Van Hiele's stages of geometric understanding also may appear to be similar to modeling cycles that emerge in some model-eliciting activities.

What's similar about the theories of both Piaget and Van Hiele is that both considered mathematical thinking to be about *seeing* (or *interpreting*) situations at least as much as it is about *doing* algorithmic procedures. Also, both focus on holistic properties of conceptual systems that students use to interpret their experiences. That is, they focus on properties of systems-as-a-whole that are not simply derived from properties of their constituent elements. For example, Piaget's *conservation tasks* all focus on children's understanding of properties that are invariant under some system of transformation. Therefore, by exhibiting an understanding of *invariance with respect to a system*, students automatically demonstrate that they are using the relevant system-as-a-whole to interpret their experiences (Lesh & Carmona, 2002).

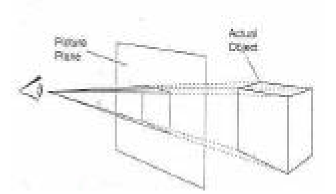
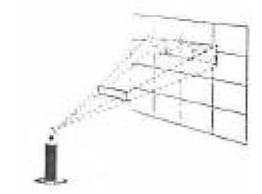
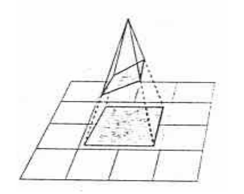
A BRIEF DESCRIPTION OF VAN HIELE'S STAGES OF DEVELOPMENT

The following stages provide one way to characterize Van Hiele's description of the development of students' geometric understandings (Van Hiele, 1959; Hoffer, 1983)

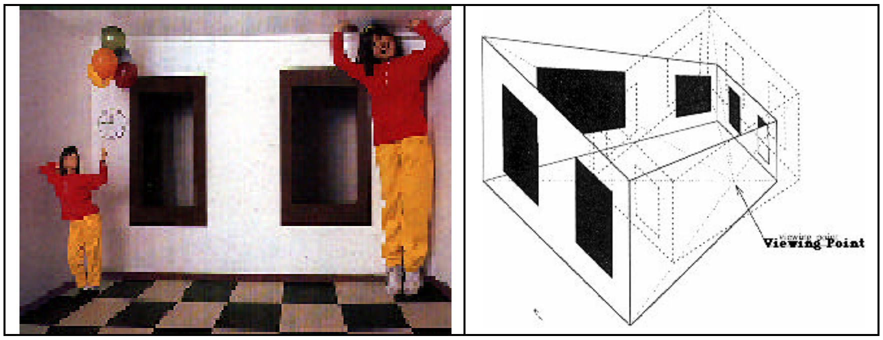
- First, geometric shapes are considered to be alike or different based on global characteristics of the shapes-as-a-whole. Thus, a rectangle (▯) might be recognized as being the same as the square (□) simply because "it looks square" to a child. Whereas, a diamond (◇) might not be recognized as being a square – simply because "it doesn't look square." … Similarly, if a young child is shown the shadow of a tilted circle (so the shadow looks like an oval), and if she is asked to "draw the shape of the shadow that you see," she often will refuse to draw anything that does not look like a circle.
- Second, geometric shapes are considered to be alike or different based on shared properties. So, a square is considered to be "a shape that four equal sides" (or four equal angles). But, a shape that HAS certain properties is not the same thing as a shape that is DEFINED IN TERMS OF these properties. … If S is the shape, and P_1, P_2, … P_n are the properties, then the distinction we're referring to is the difference between the statement *S implies* $P_1, P_2, \ldots P_n$ and the statement $P_1 + P_2 + \ldots + P_n$ *S implies S.*
- Third, geometric shapes are defined in terms of their properties, and by relationships among their properties; and, additional properties can be deduced that might not be immediately obvious by inspection.
- Forth, geometric shapes are enhanced using auxiliary elements or properties, or by embedding them within larger shapes or systems.

ANOTHER EXAMPLE OF LOCAL CONCEPTUAL DEVELOPMENT

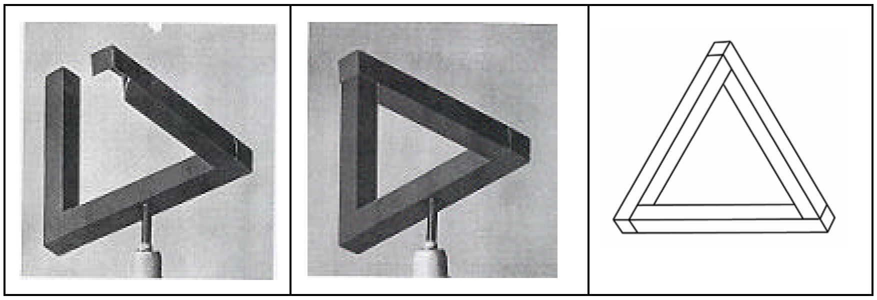
To introduce students to the *Shadows Problem*, they usually are asked to read and discuss two math-rich newspaper articles about a science fair where the exhibits will focus on the theme of *perception and illusion*. One article describes similarities between (1) photographs, (2) shadows, and (3) slices of a cone or a pyramid.

		
Photographs	Shadows	Slices from a Pyramid

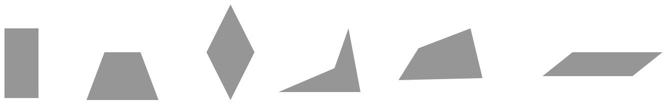
The second newspaper article described how some students were able to create illusions using shadows, photographs, or "peek holes" in distorted rooms. For example, the picture below shows how a peek-hole in a large box can be used to make two identical toy dolls appear to be completely different in size.



Similarly, the picture below shows how a peek-hole in a large box can be used to make a three dimensional shape look like a two dimensional triangle.



The problem that was given to the students told them about a group of students who were hoping to set up an exhibit in which a point source of light would be used, and non-square shapes (like the ones shown below) would be used to make square shadows. The problem statement asked the students to write a brief two-page letter to the client describing: (1) which of the following figures can be used to make a square shadow, and (2) exactly how should the figure be held, relative to the light and the wall, in order to make a square shadow. … The students were given the six cardboard shapes shown below. For a light source, they also were given a small penlight flashlight with the reflector painted black so that it will come close to providing a "point source" of light.



A complete transcript for one group of students is given at the following web site. **http://tcct.soe.purdue.edu/library/** Again, the students (Al, Bev, and Candy) whose solution to the shadows problem is described were inner city Afro-American students who were in a remedial math course for eighth graders. The session occurred in the late Fall after the students had gained experience working on four other *model-eliciting activities*. In particular, these students were experienced at working in groups, and at working independently for a full class period. Also, they were experienced at having their work videotaped; and, because their school used "block scheduling" for their classes, the class periods were twice as long as in many schools. So, on "problem solving days" the typical classroom routine was for students to work on a problem during one full class period; then, during the following class period, they'd make presentations about their work.

A Brief Summary of a Shadows Transcript

Interpretation #1 - Focusing on Unanalyzed Shapes-as-a-whole. For the first 10 minutes of the session, the students were not working well as a team. Mainly, they functioned as if they were three individuals working in the same place. There was little collaboration; and, little communication was occurring to coordinate their efforts.

- The students appeared to be thinking of shapes as-a-whole instead of analyzing them in terms of their properties. So, the only shapes that were being considered were those that seemed to "look something like squares" (i.e., the rectangle and the diamond).

- Investigations were constrained by several implicit assumptions that were not given in the problem. For example, shapes were never tilted relative to the wall, and the light was always held perpendicular to the wall.

<u>Transition Interpretation # 2 - Beginning to Notice How Movements in the Shapes Produce Changes in the Shadows.</u> Here, for the first time in the session, the students explicitly began to tilt and turn the cardboard shapes - and to carefully observe corresponding changes in the shadows that were produced. But, even though the students were beginning to pay attention to ways the shadows change when the shapes were manipulated, they were not exploring transformations in any systematic way.

<u>Interpretation #3 - Implicitly Thinking of Similarities between Vision and Shadows</u>. Without making their thinking explicit, the students were beginning to reason by analogy—implicitly using "the geometry of vision" to make predictions about "the geometry of shadows." But, the students didn't follow up on this analogy; they continued to think in terms of unanalyzed shapes-as-a-whole. For example, just like the early stages of reasoning identified by Van Hiele (1959), Bev refused to recognize a diamond as being a square – simply because its sides were not vertical and horizontal. … In general, up to this point in the session, the students seemed to be making the implicit assumption that a shape will only make a square shadow if it already "looks something like a square" by having either: (i) all of it's sides equal, or (b) all of its angles equal.

<u>Interpretation #4 - Focusing on Details as Shapes are Transformed.</u> Here, for the first time, the students clearly begin to investigate how continuous transformations in one figure lead to other kinds of continuous transformations in the shadow. For example, Al held up the rectangle and turned it very slowly in the light—while watching the shadow change. … Unlike earlier in the session, the three students are beginning to work well together well. For example, one held the light; one moved the shape; and one checked the shadow – while each was paying attention to what the others did and said.

<u>Interpretation #5 - Focusing on Relationships among *Properties* of Shapes</u>. Whereas these students' early interpretations were based on unanalyzed shapes-as-a-whole, just as in the early stages of conceptual development described by Van Hiele, they now began focusing on properties of shapes—or on shapes whose properties are recognized. Also, they explicitly began to investigate changes in properties as different shapes were transformed.

<u>Interpretation #6 Taking into Consideration the Source and Direction of the Light</u>. Up until this point in the session, the students had not consciously manipulated the direction the light was shining. They've simply shined the light perpendicular to the wall and moved the cardboard shapes. Now, they began to shine the light at an angle to the wall. That is, partly because they sometimes moved the shape and light source so that part of the shadows went beyond the side of the wall that they were using as a projection screen, the students began to hold a shape fixed and move the light source. They're also explicitly investigating how the shadow changes when the light source moves. So, they're thinking about relationships that involve changes in the light source, the cardboard shapes, and the shadow. … As they explored the preceding relationships, they were becoming conscious of increasing numbers of details about shapes, positions, movements, and shadows. For example, for the first time, the students noticed that, when the shadows get

too big, the edges of the shadow get fuzzy.

Interpretation #7A - Focusing on Transformations of Sides of Shapes, Rather than Shapes-as-a-whole. Here, for the first time, the students explicitly focused on transforming a single side of a shape. They weren't simply transforming the shapes-as-a-whole. For example, Bev focused on the longest side of the trapezoid, and she noticed that she could make the shadow of this side as long (or as short) as she wanted—by tilting it or by moving it closer to the light source. … Also, for the first time, the students explicitly investigated relationships involving *pairs* of sides – rather than simply focusing on the size of individual sides, or general shapes-as-a-whole.

Interpretation #7B – Focusing on Transformations of Sides & Angles, Rather than on Shapes-as-a-whole. This was the first time in the session that the students were explicit about paying attention to variations in angles—apart from the contributions that angles made to the general shapes-as-a-whole.

Interpretation #8A - Focusing on Complex Transformations Consisting of a Series of Simpler Transformations. Here, the students tried to produce the shadows they wanted using a series of simpler transformations – each focusing on a single attribute of the final result (rather than simply transforming shapes-as-a-whole). This is the first time that the students tried to hold one characteristic constant while varying other characteristics. For example, Al made two of the adjacent sides of the rectangle "almost the same size." Then, he tried to keep these two sides the same while tilting the rectangle to make the size of the interior angle look "more square" (i.e., 90 degrees). (note: While doing this, he didn't pay any attention to the other sides or angles.)

Interpretation #8B - A Shape is Defined by Specific Properties. At this point in the session, the students were being very careful about inspecting all of the properties of the shadow—not just one or two properties that they happened to notice. Also, the students were beginning to go beyond thinking that "if it's a square then it has properties A, B, and C" and they started thinking that "if it has properties A, B, and C, then it's a square." That is, a square was considered to be defined by its properties—rather than simply being a shape that has these properties. For example, for the first time, for an given shape, the students systematically paid attention to all four sides and all four angles at the same time.

Interpretation #9 - Going Beyond "Looking Square" to Really "Being Square". This was the first that the students seemed to recognize the need for a "proof" (or a sensible explanation) that the shadow went beyond simply "looking square" to actually "being square" (by having the necessary properties that define square-ness).

The letter shown below is a digital simulation of the one that was written by Al, Bev, and Candy with formatting and editing help from their teacher. While Bev was busy writing, Candy and Al talked (mainly "off task") – but continued to play with the shapes, light, and shadows. In addition to using the cardboard shapes, they also used their hands to make more "animal faces" and other funny shadows. Then, Al tore the diamond in half to make two equilateral triangles; and, he asked Candy "How many different triangles do you think we can make?" … They concluded that: (1) "We can make any triangle we want." (2) "We can make the angles as big or as little as we want." (3) "We can make the sides as big or as little as we want." … They demonstrated Point #1 using a broken pencil that the students had used throughout the session.

Dear Students,

You can make square shadows with these shapes.

Shapes like this one don't work.

Follow these three steps.

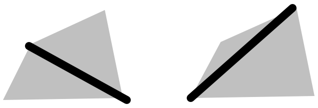
1. Make one corner square.
2. Make the two sides the same.
3. Go on around and fix each of the sides and corners.

It works. We showed it.

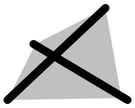
Bev, Candy, and Al

On the next day after the students had finished writing their letters to their client (who was assumed to be some students in another school), they made brief 5-minute presentations to their whole class about how they solved the *Shadows Problem.* When Bev, Cindy, and Al gave their presentation, Bev read the letter, and Candy and Al demonstrated with the light and cardboard shapes. When the teacher asked them to go beyond explaining how their method worked to also show why it worked, Al explained by tearing one of the quadrilaterals in half to make two triangles. Then, he showed how to make "half of a square" with this triangle. Finally, he drew a sketch, like the one shown below, to show: (1) how any of the convex quadrilaterals can be cut be cut into two triangles, and (2) how any of these triangles could be used to make "half of a square shadow". After all the students in the class had made presentations about their work, Al, Bev, and Candy revised their letter to include the four steps shown below.

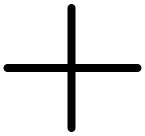
1. Take the shape and cut it into two triangles. You can do it two ways.



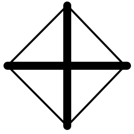
2. Use straws to make kite pieces.



3. Use only the kite pieces and tilt them to make a square shadow. It's easy.



4. Put the paper back on the kite and the shadow will look square too.



Interpretation #10 - Using Auxiliary Lines & Shapes, Implicitly Introducing Movements in Three Dimensions, & Providing an Informal Proof. Whereas the students' first letter described a method that required a lot of tinkering to make it work, their second letter constituted an actual informal proof that "a square shadow can be made using any convex quadrilateral." The procedure they specified used the kind of auxiliary lines (and trangles) shown in the diagrams above. Also, the students implicitly described movements in three dimensions. That is, they slid the shape in one direction; they turned it in a second direction; and finally, they turned it in a third direction.

CONCLUSIONS

For each of the four transcripts described in this paper, our analyses focused on the modeling cycles that students went through during a ninety-minute problem solving session. In other publications, and for other purposes, we've focused on other ways of thinking about students' behaviors. For example, Jawojewski & Lesh (2002) analyze similar transcripts by focusing on the problem solving strategies and heuristics that the students employed; Jawojewski, English, & Lesh (2002) focus on changes in group functioning throughout solution processes; Middleton, Lesh, & Heger (2002) focus on social and affective factors that impact students behaviors; and, Harel & Lesh (2002) focus on proof strategies for the same *Shadows Transcript* that we described here. … Readers may wish to use still other perspectives to analyze the transcripts we've described. Complete transcripts can be downloaded from the following web site. **http://tcct.soe.purdue.edu/library/**

For the purposes of this paper, the first point that we want to emphasize is that, when a *model-eliciting activity* requires students to develop a conceptual tool that involves a construct or conceptual system whose stages of development have been investigated by developmental psychologists or mathematics educators, then the task-specific modeling cycles often bear striking similarities to corresponding general stages in the development. For example, both Piaget's and Van Hiele's theories of conceptual development can be used to help understand and explain many of the behaviors student exhibit during the process of developing conceptual tools needed in *model-eliciting activities*. Consequently, it is reasonable to expect that processes that contribute to general cognitive development also should contribute to progress through the modeling cycles for *model-eliciting problems*.

A second related cluster of implications is related to the fact that, if we examine a student's performances across a series of related activities, it's clear that his or her apparent stage of development often varies considerably across tasks as well as across modeling cycles for a specific task. This is significant for a variety of reasons. For example, whereas Piaget inspired researchers often speak of mathematical constructs (such as those related to fractions, ratios, rates, proportions) as if they were specific manifestations of general/all-purpose cognitive structures, the *local conceptual developments* that we observe during *model-eliciting activities* suggest that the evolution of mathematical constructs tends to be more accurately characterized as gradual increases in local competence. That is, relevant conceptual systems are developed first as situated models that apply to particular problem solving situations. Then, these models are gradually extended to larger classes of problems as they become more sharable, more transportable, and more reusable. In other words, it is only at relatively mature levels of thinking that knowledge begins to be organized around abstractions rather than around experiences; and, these mature levels of thinking are not likely to evolve unless: (i) students are challenged to develop models and conceptual tools that are sharable, re-useable, and transportable, (ii) students are introduced to powerful representation systems for expressing relevant constructs, and (iii) students are encouraged to go beyond thinking with these constructs to also think about them. … On the other hand, if these latter functions are emphasized, then significant generalizations often can be expected from one task to other similar tasks. Therefore, when we speak about *local conceptual development*, we are not suggesting that all learning is task specific—nor that there is no generalization or transfer of learning. In fact, by emphasizing the development of conceptual tools that are sharable, re-usable, and transportable, it is much more likely that the relevant constructs and conceptual systems will be useful beyond the situations in which they were created. For example, consider the simple case when a problem is solved using spreadsheet generated graphs. As anybody knows who has used such tools, some spreadsheets are easier than others to adapt to new data, new purposes, or new situations. Yet, even if the spreadsheet is designed to facilitates transfer, the students who created the spreadsheet may or may not be able to use to tool in a new situation problem solving situation.

A final point becomes apparent when we examine details within transcripts where this brief report was only able to describe general stages in development. That is, contrary to the ladder-like descriptions of development that stages suggest, details in the transcripts show that students often switched back and forth among a variety of different ways of thinking about a problem solving situations—usually without noticing that these conceptual shifts had been made. In fact, even within the thinking of individual students, communities of constructs and conceptual systems often were competing to dominate the interpretations that would be emphasized at any given moment. Thus, at any time during a series of modeling cycles, "communities of mind" were apparent not only within the thinking of teams but also within the thinking of individuals within the teams. So, solution processes often involved gradually differentiating and/or integrating (or dismissing, or refining) alternative ways of thinking—rather than simply progressing along ladder-like sequences. … In other words, regardless what developmental theory is used to explain students' performances during *model-eliciting activities*, it tends to be far too simplistic to refer to a given child as being at stage N in some developmental sequence. Communities of relevant concepts tend to be available at any given moment; most of these constructs are at intermediate stages of development, and apparent levels of development vary across tasks as well as across time within a given task.

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